

# Curves of maximal genus in $\mathbb{P}^5$

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## 1 Introduction

Let  $C \subset \mathbb{P}^r$  be a reduced, irreducible, not degenerate curve, not contained on surfaces of degree  $< s$ ; when  $d = \deg C$  is large with respect to  $s$ , the *arithmetic* genus  $p_a(C)$  is bounded by a function  $G(d, r, s)$  which is of type  $\frac{d^2}{2s} + O(d)$ .

The existence of such a bound for  $C \subset \mathbb{P}^3$  was announced by Halphen in 1870 and proved by Gruson and Peskine in [GP]; for curves in  $\mathbb{P}^r$ ,  $r \geq 4$  the bound is stated and proved in [CCD] ( for  $d > \frac{2s}{r-2} \prod_{i=1}^{r-2} ((r-1)!s)^{\frac{1}{r-1-i}}$ ).

The existence of curves of maximal genus, i.e. whose genus attains the bound, is known in  $\mathbb{P}^3$  for  $d > s^2 - s$ , in  $\mathbb{P}^4$  for  $d > 12s^2$  and in  $\mathbb{P}^r$ ,  $r \geq 5$ , at least for  $d \gg s$ . [GP] contains a precise description of those curves which do not lie on surfaces of degree  $< s$  and whose genus is maximal: they are arithmetically Cohen-Macaulay, lie on a surface  $S$  of degree  $s$  and they are directly linked to plane curves. [CC] contains the description of curves in  $\mathbb{P}^4$  of maximal genus  $G(d, 4, s)$ .

The complete description of curves  $C \subset \mathbb{P}^5$ , not contained on surfaces of degree  $< s$ , whose genus is  $G(d, 5, s)$  has been given by the author in her PhD dissertation [F1]. The main result of this note is the classification Theorem 1.1, which holds for  $s \geq 9$ . Due to the long list of cases, some proofs are given only in some specific examples. We already know [CCD] that such curves must be arithmetically Cohen-Macaulay and they must lie on a surface  $S$  of degree  $s$ , whose general hyperplane section  $\Gamma$  is a "Castelnuovo curve" in  $\mathbb{P}^4$ , i.e. a curve in  $\mathbb{P}^4$  of maximal genus. When  $s \geq 9$  the surface  $S$  lies on a rational normal 3-fold  $X$  of degree 3 in  $\mathbb{P}^5$ , which can be singular. Analogously to [GP] and [CC], we describe our curves  $C$  of genus  $G(d, 5, s)$  in terms of the curve  $C'$  obtained by linking  $C$  with  $S$  and a hypersurface  $F$  of minimal degree passing through  $C$  and not containing  $S$ .

In [dC], the author derives an upper bound for the *geometric* genus of integral curves on the three dimensional *nonsingular* quadric which lie on an integral surface of degree  $2k$ , as a function of  $k$  and the degree  $d$  of the curve, without any assumption on the degree  $d$ . The author analyzes the curves that should achieve that bound, which turns out to be *not sharp*.

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We state now the main Theorem. In Propositions 4.1, 4.2 and 4.3 we will give a closer description of cases 2) 3) and 4) of the Theorem.

**Theorem 1.1** *Let  $C \subset \mathbb{P}^5$  be an integral non degenerate curve not contained on surfaces of degree  $< s$  and let  $s \geq 9$ ; put  $d = \deg C$  and let  $p_a(C) = G(d, 5, s)$  be its arithmetic genus. Assume  $d > \frac{2s}{3} \prod_{i=1}^3 (4!s)^{\frac{1}{4-i}}$ .*

*Then  $C$  is arithmetically Cohen-Macaulay and lies on a irreducible surface  $S$  of degree  $s$  contained in cubic a rational normal 3-fold  $X \subset \mathbb{P}^5$ .*

*Put  $d - 1 = sm + \epsilon$ ,  $0 \leq \epsilon \leq s - 1$  and  $s - 1 = 3w + v$ ,  $v = 0, 1, 2$ ;*

*if  $\epsilon < w(4 - v)$ , divide  $\epsilon = kw + \delta$ ,  $0 \leq \delta < w$ ;*

*if  $\epsilon \geq w(4 - v)$ , divide  $\epsilon + 3 - v = k(w + 1) + \delta$ ,  $0 \leq \delta < w + 1$ .*

*Then  $C$  is contained on a hypersurface  $F$  of degree  $m + 1$ , not passing through  $S$ . If  $C'$  is the curve linked to  $C$  by  $F$  and  $S$ , we have:*

- 1. when  $k = 3$ , then  $C' = \emptyset$ ; i.e.  $C$  is complete intersection on  $S$ ;*
- 2. when  $k = 2$ , then  $C'$  is a plane curve;*
- 3. when  $k = 1$ , then  $C'$  lies on a surface of degree 2;*
- 4. when  $k = 0$ , then  $C'$  lies on a surface of degree 3.*

*Finally, for all  $s, d$  with  $s \geq 4$  and  $d > \frac{2s}{3} \prod_{i=1}^3 (4!s)^{\frac{1}{4-i}}$ , one can find a smooth curve of degree  $d$ , arithmetic genus  $G(d, 5, s)$  which does not lie on surfaces of degree  $< s$ .*

The proof is based on the analysis of the Hilbert function of a general hyperplane section  $Z$  of  $C$ . The main technical problem that one does not find in the previous cases ( $r = 3, 4$ ) is that for describing  $C'$  we have to perform a linkage by a complete intersection on the scroll  $X$ , which is in general, if  $X$  is singular, a non-Gorenstein scheme. To this purpose the author has proved in [F2] and [F3] some general results to which we will refer in these note.

The last section is devoted to examples. We produce smooth curves of maximal genus for all  $s \geq 4$  and  $d > \frac{2s}{3} \prod_{i=1}^3 (4!s)^{\frac{1}{4-i}}$ . It should be observed that in [CC] the authors don't analyze the regularity of the produced extremal curves in  $\mathbb{P}^4$  and that in [CCD] the produced examples of curves of maximal genus  $G(d, r, s)$  in  $\mathbb{P}^r$  for  $d \gg s$  are in general singular.

With the same techniques used for the classification in  $\mathbb{P}^5$  it is possible to classify curves in  $\mathbb{P}^r$  of maximal genus  $G(d, r, s)$  for every  $r$  and  $s \geq 2r - 1$ . In [F3] the author has given an example of the classification procedure for curves of maximal genus  $G(d, r, s)$  in  $\mathbb{P}^r$  and of the construction of such smooth extremal curves.

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## 2 Weil divisors on $X$

We will see in the next section that curves  $C \subset \mathbb{P}^5$  of maximal genus  $G(d, 5, s)$  which we want to classify lie on a cubic rational normal 3-fold  $X \subset \mathbb{P}^5$  about which we need to fix some notation and mention some result. A rational normal 3-fold  $X \subset \mathbb{P}^5$  is the image of a projective bundle  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$  over  $\mathbb{P}^1$ , via the morphism  $j$  defined by the tautological line bundle  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ , where  $\mathcal{E}$  is a locally free sheaf of rank 3 on  $\mathbb{P}^1$  of one of the following three kinds:

1.  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . In this case  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$  is smooth and it is ruled by  $\infty^1$  disjoint planes; we put  $X = S(1, 1, 1)$ .
2.  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ . Here  $X$  is a cone over a smooth cubic surface in  $\mathbb{P}^4$  with vertex a point  $V$  and it is ruled by  $\infty^1$  planes intersecting at  $V$ ; we put  $X = S(0, 1, 2)$ .
3.  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ .  $X$  is a cone over a twisted cubic in  $\mathbb{P}^3$  with vertex a line  $l$ , and it is ruled by  $\infty^1$  planes intersecting at  $l$ ; we put  $X = S(0, 0, 3)$ .

Let us denote  $\mathbb{P}(\mathcal{E}) = \tilde{X}$ . The morphism  $j : \tilde{X} \rightarrow X$  is a rational resolution of singularities, called the *canonical resolution* of  $X$ . It is well known that the Picard group  $\text{Pic}(\tilde{X})$  of  $\tilde{X}$  is isomorphic to  $\mathbb{Z}[\tilde{H}] \oplus \mathbb{Z}[\tilde{R}]$ , where  $[\tilde{H}] = [\mathcal{O}_{\tilde{X}}(1)]$  is the hyperplane class and  $[\tilde{R}] = [\pi^* \mathcal{O}_{\mathbb{P}^1}(1)]$  is the class of the fiber of the map  $\pi : \tilde{X} \rightarrow \mathbb{P}^1$ . The intersection form on  $\tilde{X}$  is determined by the rules:

$$\tilde{H}^3 = 3 \quad \tilde{R} \cdot \tilde{H}^2 = 1 \quad \tilde{R}^2 \cdot \tilde{H} = 0. \quad (2.1)$$

The cohomology of the invertible sheaf  $\mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R})$  associated to a divisor  $\sim a\tilde{H} + b\tilde{R}$  in  $\tilde{X}$  can be explicitly calculated using the Leray spectral sequence. In particular, for  $a \geq 0$  and  $b \geq -1$ , the dimension  $h^0(\mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R}))$  does not depend on the type of the scroll and it is given by the formula ([F2] 3.5):

$$h^0(\mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R})) = 3 \binom{a+2}{3} + (b+1) \binom{a+2}{2}. \quad (2.2)$$

Let  $H$  and  $R$  be the *strict images* in  $X$  of  $\tilde{H}$  and  $\tilde{R}$  respectively, i.e. the scheme-theoretic closures  $\overline{j(\tilde{H}|_{j^{-1}X_S})}$  and  $\overline{j(\tilde{R}|_{j^{-1}X_S})}$ , where  $X_S$  denotes the smooth part of  $X$ . Let us consider on  $X$  the direct image of  $\mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R})$  in  $X$  for every  $a, b \in \mathbb{Z}$ :

$$\mathcal{O}_X(aH + bR) := j_* \mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R}).$$

If the scroll  $X$  is smooth, then the sheaves  $\mathcal{O}_X(aH + bR)$  are the invertible sheaves associated to the Cartier divisors  $\sim aH + bR$ , while when  $X$  is singular this is no longer true. In this case we have the following Proposition which has been proved in [F2] (Lemma 2.14, Cor. 3.10 and Th. 3.17). The reader may refer to [H1] for a survey on *divisorial sheaves* associated to generalized divisors, in particular to Weil divisors.

**Proposition 2.3** *Let  $X \subset \mathbb{P}^5$  be a rational normal 3-fold and let  $j : \tilde{X} \rightarrow X$  be its canonical resolution. Let  $\text{Cl}(X)$  be the group of Weil divisors on  $X$  modulo linear equivalence. Then*

- *If  $X$  is smooth or  $X = S(0, 1, 2)$  we have  $\text{Cl}(X) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[R]$ . The divisorial sheaf associated to a divisor  $\sim aH + bR$  on  $X$  is  $\mathcal{O}_X(aH + bR)$  for every  $a, b \in \mathbb{Z}$ .*
- *If  $X = S(0, 0, 3)$  we have that  $H \sim 3R$  and  $\text{Cl}(X) \cong \mathbb{Z}[R]$ . The sheaves  $\mathcal{O}_X(aH + bR)$  with  $a + 3b = d$  and  $b < 3$  are all isomorphic to the divisorial sheaf associated to a divisor  $\sim dR$ .*

**Remark** *Since  $R^i j_* \mathcal{O}_{\tilde{X}}(a\tilde{H} + b\tilde{R}) = 0$  for  $i > 0$  and for all  $a \in \mathbb{Z}$  and  $b \geq -1$ , then formula (2.2) holds for  $h^0(\mathcal{O}_X(aH + bR))$  too. In particular we can use formula (2.2) to compute  $\dim |D|$  for an effective divisor  $D \sim aH + bR$  in  $X = S(1, 1, 1)$  or  $X = S(0, 1, 2)$  with  $b \geq -1$ . When  $X = S(0, 0, 3)$  by Prop. 2.3 we can write the divisorial sheaf associated to  $D \sim dR$  in the form  $\mathcal{O}_X(aH + bR)$  with  $0 \leq b < 3$ ; therefore we can use formula (2.2) to compute  $\dim |D|$  for every effective divisor  $D$  in  $X = S(0, 0, 3)$ .*

When  $X$  is smooth or  $X = S(0, 1, 2)$  the intersection form (2.1) on  $\tilde{X}$  determines via the isomorphism  $\text{Pic}(\tilde{X}) \cong \text{Cl}(X)$  of Prop. 2.3 the intersection form on  $X$ . In particular we use the intersection number  $D \cdot D' \cdot H$  of two effective divisor  $D \sim aH + bR$  and  $D' \sim a'H + b'R$  with no common components to compute the degree of their scheme-theoretic intersection  $D \cap D'$ .

When  $X = S(0, 0, 3)$  the computation of  $\deg(D \cap D')$  is more complicate and it is explained in detail in [F2]; here we want just to state the results that we need. So let us first introduce the *integral total transform* of an effective Weil divisor  $D \sim dR$ :

**Definition 2.4** *Let  $X = S(0, 0, 3)$ . Let  $D \subset X$  be an effective Weil divisor. Then the integral total transform  $D^*$  of  $D$  in  $\tilde{X}$  is:*

$$D^* := \tilde{D} + [q]E$$

where  $\tilde{D} \sim a\tilde{H} + b\tilde{R}$  is the proper transform of  $D$  in  $\tilde{X}$ ,  $E \sim \tilde{H} - 3\tilde{R}$  is the exceptional divisor in  $\tilde{X}$  and  $[q]$  is the smallest integer  $\geq q := \frac{b}{3}$ .

Then let us introduce, for every effective Weil divisor  $D$  in  $X$ , the rational number  $\epsilon := [q] - q$ . We can compute the degree of the intersection scheme  $D \cap D'$  of two effective divisors  $D \sim dR$  and  $D' \sim d'R$  with no common components using the following formula which has been proved in [F2] Prop. 4.11:

$$\deg(D \cap D') = \begin{cases} D^* \cdot D'^* \cdot \tilde{H} & \text{if } [\epsilon + \epsilon'] = 0 \\ D^* \cdot D'^* \cdot \tilde{H} + 3(\epsilon + \epsilon' - 1) + 1 & \text{if } [\epsilon + \epsilon'] = 1. \end{cases} \quad (2.5)$$

By abusing notation we will write the degree  $\deg(D \cap D')$  by the intersection number  $D \cdot D' \cdot H$ . Moreover we compute the intersection multiplicity  $m(D, D'; l)$  of  $D$  and  $D'$  along the singular line  $l$  of  $X$  as follows:

$$m(D, D'; l) = D \cdot D' \cdot H - \tilde{D} \cdot \tilde{D}' \cdot \tilde{H}. \quad (2.6)$$

**Remark** *To prepare the proof of the main Theorem and of Propositions 4.1, 4.2, 4.3 we briefly describe all possible planes and surfaces of degree 2 and 3 contained in a rational normal 3-fold  $X \subset \mathbb{P}^5$ .*

1. When  $X = S(1, 1, 1)$ . The only planes contained in  $X$  are the ones in the ruling of  $X$ , otherwise the linear system  $|\mathcal{O}_X(R)|$  would cut on a plane  $\pi \subset X$ , which is not in the ruling, a pencil of lines which of course intersect each other, while the planes in  $|\mathcal{O}_X(R)|$  are pairwise disjoint. The surfaces of degree 2 contained in  $X$  are either reducible and hence linearly equivalent to  $2R$ , or irreducible and hence degenerate, therefore linearly equivalent to  $H - R$ . A reduced surface  $Y \sim 2R$  is the disjoint union of two planes in  $\mathbb{P}^5$ . A surface  $Q \sim H - R$  is a smooth quadric surface; the two systems of lines on  $Q$  are cut by the linear system  $|\mathcal{O}_X(R)|$  (lines of type  $(1, 0)$  on  $Q$ ) and by the linear system  $|\mathcal{O}_X(H - R)|$  (lines of type  $(0, 1)$  on  $Q$ ). The surfaces of degree 3 contained in  $X$  are either reducible in the union of three planes and hence linearly equivalent to  $3R$ , or reducible in the union of an irreducible quadric surface and of a plane, hence linearly equivalent to  $H - R + R \sim H$ , or finally irreducible, therefore degenerate, and so linearly equivalent to an hyperplane section  $H$ . A reduced surface  $\sim 3R$  is the disjoint union of three planes in  $\mathbb{P}^5$ . A reducible hyperplane section of  $X$  is the union of a smooth quadric surface and of a plane meeting along a line of type  $(1, 0)$ . An irreducible hyperplane section of  $X$  is a smooth rational normal surface in  $\mathbb{P}^4$ . Lastly we recall that a surface  $\sim aH + bR$  on  $X$  is irreducible when  $a = 0$  and  $b = 1$ , or  $a > 0$  and  $b \geq -a$  (by [H2], V, 2.18, passing to general hyperplane sections).
2. When  $X = S(0, 1, 2)$ , a plane contained in  $X$  is either one of the ruling of  $X$ , therefore linearly equivalent to  $R$ , or it is the plane  $p \sim H - 2R$ , i.e. the plane spanned by the vertex  $V$  of  $X$  and by the line image of the section defined by  $\text{Proj } \mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow \mathbb{P}(\mathcal{E})$ . The reducible surfaces of degree 2 contained in  $X$  are either linearly equivalent to  $2R$  (when reduced they are the union of two planes meeting at the point  $V$ ), or linearly equivalent to  $H - R$  (the union of  $p$  and of a plane  $\pi \sim R$  meeting along a line passing through  $V$ ), or linearly equivalent to  $2(H - 2R)$  (the plane  $p$  counted with multiplicity 2). The irreducible ones are linearly equivalent to  $H - R$ . An irreducible surface  $Q \sim H - R$  is a quadric cone with vertex  $V$ ; the pencil of lines on  $Q$  is cut by the linear system  $|\mathcal{O}_X(R)|$  (or equivalently by the linear system  $|\mathcal{O}_X(H - R)|$ ). The surfaces of degree 3 contained in  $X$  are either reducible in the union of three planes and hence linearly equivalent to  $3R$  (when reduced they are the union of three planes

meeting at the point  $V$ ), or to  $2R + H - 2R \sim H$  (when reduced each plane  $\sim R$  meets the plane  $p$  along a line passing through the point  $V$ ), or to  $R + 2(H - 2R) = 2H - 3R$ , or to  $3(H - 2R)$  (in the last two cases the surface is not reduced). They may be also reducible in the union of an irreducible quadric cone and of a plane, hence linearly equivalent to  $H - R + R \sim H$  (the cone and the plane meet along a line passing through  $V$ ), or to  $H - R + H - 2R = 2H - 3R$  (the cone and the plane  $P$  meet at the point  $V$ ). Finally they can be irreducible, therefore degenerate and so linearly equivalent to an hyperplane section  $H$ , which is a rational normal surface in  $\mathbb{P}^4$ . As in the previous case, since a general hyperplane section of  $X$  is smooth, we have that a surface  $\sim aH + bR$  on  $X$  is irreducible when  $a = 0$  and  $b = 1$ , or  $a > 0$  and  $b \geq -a$ .

3. When  $X = S(0, 0, 3)$  the situation is simpler since  $\text{Cl}(X) = \mathbb{Z}[R]$ . A plane contained in  $X$  is a plane of the ruling. A surface of degree 2 contained in  $X$  is always linearly equivalent to  $2R$  and it is always reducible in the union of two planes meeting along the line  $l$  (the vertex of  $X$ ). A surface of degree 3 contained in  $X$  is linearly equivalent to  $3R$  and it is reducible in the union of three planes meeting at  $l$  if the proper transform is linearly equivalent to  $3\tilde{R}$ , or it is irreducible, and hence a singular rational normal surface in  $\mathbb{P}^4$ , if the proper transform is linearly equivalent to  $\tilde{H}$ . In this case, since the a hyperplane section of  $X$  is singular, by [H2], V, 2.18 we have that a surface  $S \subset X$  is irreducible if its proper transform is  $\tilde{S} \sim a\tilde{H} + b\tilde{R}$ , with  $a = 0$  and  $b = 1$ , or  $a > 0$  and  $b \geq 0$ .

**Remark** *Finally we want to describe how plane curves contained in  $X$  look like. Since  $X$  is intersection of quadrics in  $\mathbb{P}^5$ , then a plane curve of degree  $\geq 3$  contained in  $X$  must lie in a plane of  $X$ . Therefore we are interested just in lines and conics.*

1. Let  $X = S(1, 1, 1)$ . A line  $r \subset X$  which is not contained in a plane  $\pi \sim R$  of the scroll is the base locus of a pencil of quadric surfaces  $\sim H - R$ , i.e. it is a line of type  $(0, 1)$  (take the pencil of hyperplane sections passing through  $r$  and a fixed plane  $\sim R$  intersecting  $r$ ). A conic  $C \subset X$  which does not lie on  $\pi \sim R$  is contained in a quadric surface  $Q \sim H - R$  (take a hyperplane section passing through  $C$  and a plane  $\sim R$  meeting  $C$ ). Therefore it is an hyperplane section of  $Q$ , i.e. a curve of type  $(1, 1)$  on  $Q$ .
2. Let  $X = S(0, 1, 2)$ . Every line  $r \subset X$  is contained in a plane of the scroll. In fact if  $r$  passes through  $V$ , then it is obviously contained in some plane  $\pi \sim R$ . If  $r$  does not pass through  $V$ , then the plane spanned by  $r$  and  $V$  is contained in  $X$ . A conic  $C \subset X$  which is not contained in a plane of the scroll and that passes through  $V$  is reducible in the union of two lines. If  $C$  does not pass through  $V$ , then the cone over  $C$  with vertex  $V$  is a quadric cone  $Q \sim H - R$ , therefore  $C$  is an hyperplane section of  $Q$ .

3. Let  $X = S(0, 0, 3)$ . A line  $r \subset X$  is always contained in a plane of the scroll  $\pi \sim R$ . In fact it is contained in the hyperplane section passing through  $r$  and the singular line  $l$  of  $X$ , which splits in the union of three planes  $\sim R$ . A conic  $C \subset X$  which does not lie on  $\pi \sim R$  is an hyperplane section of a surface  $\sim 2R$ , i.e. it is the union of two lines meeting at a point.

### 3 Preliminaries

We start by recalling a few results of [CCD].

From now on, let  $C$  be an integral, nondegenerate curve of degree  $d$  and arithmetic genus  $p_a(C)$  in  $\mathbb{P}^5$ , with  $d > \frac{2s}{3} \prod_{i=1}^3 (4!)^{\frac{1}{4-i}}$ . Assume  $C$  not contained on surfaces of degree  $< s$  ( $s \geq 4$ ) and define  $m, \epsilon, w, v, k, \delta$  as in the statement of Theorem 1.1.

Then the genus  $p_a(C)$  is bounded by the function:

$$G(d, 5, s) = 1 + \frac{d}{2}(m + w - 2) - \frac{m+1}{2}(w - 3) + \frac{vm}{2}(w + 1) + \rho$$

where  $\rho = \frac{\delta}{2}(w - \delta)$  if  $\epsilon < w(4 - v)$  and  $\rho = \frac{\epsilon}{2} - \frac{w}{2}(3 - v) - \frac{\delta}{2}(w - \delta + 1)$  if  $\epsilon \geq w(4 - v)$  ([CCD], section 5).

If  $Z$  is a general hyperplane section of  $C$  and  $h_Z$  is the Hilbert function of  $Z$ , then the difference  $\Delta h_Z$  must be bigger than the function  $\Delta h$  defined by:

$$\Delta h(n) = \begin{cases} 0 & \text{if } n < 0 \\ 3n + 1 & \text{if } 0 \leq n \leq w \\ s & \text{if } w < n \leq m \\ s + k - 3(n - m) & \text{if } m < n \leq m + \delta \\ s + k - 3(n - m) - 1 & \text{if } m + \delta < n \leq m + w + e \\ 0 & \text{if } n > m + w + e \end{cases}$$

where  $e = 0$  if  $\epsilon < w(4 - v)$  and  $e = 1$  otherwise.

**Proposition 3.1** *If  $p_a(C) = G(d, 5, s)$ , then  $\Delta h_Z(n) = \Delta h(n)$  for all  $n$  and  $C$  is arithmetically Cohen-Macaulay. Moreover  $Z$  is contained on a reduced curve  $\Gamma$  of degree  $s$  and maximal genus  $G(s, 4) = \frac{w(w-1)(w-2)}{2} + wv$  in  $\mathbb{P}^4$  (Castelnuovo's curve). Since  $d > s^2$ ,  $\Gamma$  is unique and, when we move the hyperplane, all these curves  $\Gamma$ 's patch together giving a surface  $S \subset \mathbb{P}^5$  of degree  $s$  through  $C$ .*

**Proof.** See [CCD] 0.1, 6.1, 6.2, 6.3.  $\square$

$S$  is a "Castelnuovo surface" in  $\mathbb{P}^5$ , i.e. a surface whose general hyperplane section is a curve of maximal genus in  $\mathbb{P}^4$ .

**Proposition 3.2**  *$S$  is irreducible and when  $s \geq 9$  lies on a cubic rational normal 3-fold  $X$  in  $\mathbb{P}^5$  where it is cut by a hypersurface  $G$  of degree  $w + 1$ . As a divisor on  $X$  the surface  $S$  is linearly equivalent to  $(w + 1)H - (2 - v)R$  (or  $wH + R$  if  $v = 0$ ).*

**Proof.**  $S$  is irreducible because  $C$  is irreducible and is not contained on surfaces of degree  $< s$ .

A general hyperplane section  $\Gamma$  of  $S$  is a special Castelnuovo's curve in  $\mathbb{P}^4$  of degree  $s$ , then it lies on a rational normal cubic surface  $W$  in  $\mathbb{P}^4$  which is intersection of the quadric hypersurfaces containing  $\Gamma$ , hence also  $Z$ ; since  $C$  is arithmetically Cohen-Macaulay these quadrics must lift to quadric hypersurfaces in  $\mathbb{P}^5$  containing  $C$ , hence also  $S$ . The intersection of these quadric hypersurfaces is a rational normal cubic 3-fold  $X$  in  $\mathbb{P}^5$  whose general hyperplane section is  $W$ .

Moreover  $\Gamma$  lies on a hypersurface of degree  $w + 1$  which does not contain  $W$ ; such a hypersurface must lift to a hypersurface  $G$  of degree  $w + 1$  in  $\mathbb{P}^5$ , containing  $C$ , hence containing  $S$  since  $d > s^2$ , and not containing  $X$ .  $\square$

**Proposition 3.3** *There exists a hypersurface  $F$  of degree  $m+1$ , passing through  $C$  and not containing  $S$ .*

**Proof.** For a general hyperplane section  $\Gamma$  of  $S$ , the Hilbert function  $h_\Gamma$  is known (see e.g. [Ha2]); in particular we have  $\Delta h_\Gamma(n) = \Delta h_Z(n)$  when  $0 \leq n \leq m$  and hence  $h^0(\mathcal{I}_C(n)) = h^0(\mathcal{I}_S(n))$  when  $0 \leq n \leq m$ . For  $n = m + 1$  one computes  $\Delta h_Z(m + 1) < \Delta h_\Gamma(m + 1)$  and this implies  $h^0(\mathcal{I}_C(m + 1)) > h^0(\mathcal{I}_S(m + 1))$ .  $\square$

We recall here the definition of geometrical linkage.

**Definition 3.4** *Let  $Y_1, Y_2, Y$  be subschemes of a projective space  $\mathbb{P}$ , then  $Y_1$  and  $Y_2$  are geometrically linked by  $Y$  if*

1.  $Y_1$  and  $Y_2$  are equidimensional, have no embedded components and have no common components
2.  $Y_1 \cup Y_2 = Y$ , scheme theoretically.

**Definition 3.5** *Call  $C'$  the curve residual to  $C$  on  $S$  by  $F$ ;  $\deg C' = s - \epsilon - 1$ . Call  $C''$  the curve residual to  $C$  on  $X$  by  $F$  and  $G$ .*

We note that  $C'$  is well defined since  $S$  is irreducible and  $F$  does not contain  $S$ ,  $\deg C' = s(m+1) - d = s - \epsilon - 1$ . Moreover since  $\deg C' < \deg C$  the curve  $C'$  does not contain  $C$ , which is irreducible; therefore  $C$  and  $C'$  are geometrically linked by  $S \cap F$ . Also  $C''$  is well defined and  $C' \subset C''$ :

- if  $s = 3w + 3$  ( $v = 2$ ), then

$$C'' = C'.$$

- if  $s = 3w + 2$  ( $v = 1$ ), we can choose the plane  $p_1 \sim R$  linked to  $S$  on  $X$  by  $G$  such that it is not contained in  $F$ , then

$$C'' = C' + C_1$$

where  $C_1 \subset p_1$  is a plane curve of degree  $m + 1$ .



- if  $s = 3w + 1$  ( $v = 0$ ) and  $S \sim (w + 1)H - 2R$ , we can choose the divisor  $\sim 2R$  linked to  $S$  on  $X$  by  $G$  such that it is the union of two distinct plane  $p_1$  and  $p_2$  not contained in  $F$ , then

$$C'' = C' + C_1 + C_2$$

where  $C_1 \subset p_1$  and  $C_2 \subset p_2$  are two plane curves of degree  $m + 1$ .

- if  $s = 3w + 1$  and  $S \sim wH + R$ , we can choose the divisor  $q \sim H - R$  linked to  $S$  by  $X$  and  $G$  such that it is not contained in  $F$ , then

$$C'' = C' + C_q$$

where  $C_q$  is the intersection of  $q$  and  $F$ .

$C$  and  $C''$  are geometrically linked by  $X \cap F \cap G$  since they are equidimensional have no common components ( $C$  is irreducible and  $C'$  does not contain  $C$ ) and with no embedded components ( $X \cap F \cap G$  is arithmetically Cohen Macaulay).

**Definitions 3.6** Let  $X$  be a rational normal scroll in  $\mathbb{P}^5$ . Call  $A = Q_1 \cap Q_2$  a generic complete intersection of two quadrics in  $\mathbb{P}^5$  containing  $X$ . Call  $B \cong \mathbb{P}^3$  the linked scheme to  $X$  by  $A$ . Call  $Y_B = X \cap B$  the intersection scheme between  $X$  and  $B$ . Call  $C'''$  the curve linked to  $C$  on  $A$  by  $F$  and  $G$ .

**Remark** Hartshorne's Connectedness Theorem ([E2], Th. 18.12) implies that  $Y_B$  is a divisor; moreover, since  $Y_B \subset B$ , it follows that  $Y_B$  is contained in a hyperplane section and has degree 2 (it is cut on  $B$  by a quadric), therefore  $Y_B \sim H - R$  as a divisor in  $X$ .

**Definition 3.7** Call  $Z, Z', Z'', Z''', W$ , general hyperplane sections of  $C, C', C'', C''', X$  respectively. By abusing notation call  $A \subset \mathbb{P}^4$  and  $Y_B \subset W$  general hyperplane sections of  $A \subset \mathbb{P}^5$  and  $Y_B \subset X$  respectively.

Using the linkage techniques developed in [F3] we can prove the following results which are the main tool in the classification procedure.

**Lemma 3.8** If  $X$  is smooth or  $X = S(0, 1, 2)$ , then for  $i \leq w$ :

$$\begin{aligned} h^0(\mathcal{I}_{C''|X}(iH + R)) &\geq h^0(\mathcal{I}_{Z'''|W}(iH + R)) = \\ &= h^1(\mathcal{I}_{Z|W}(m + w - i)) = \sum_{r=m+w-i+1}^{\infty} \Delta h(r). \end{aligned}$$

Moreover if  $h^0(\mathcal{I}_{Z''|W}((i - 1)H + R)) = 0$  and  $h^0(\mathcal{I}_{Z''|W}(iH + R)) = h > 0$ , then  $h^0(\mathcal{I}_{C''|X}((i - 1)H + R)) = 0$  and  $h^0(\mathcal{I}_{C''|X}(iH + R)) = h$ .

**Proof.** Since  $C$  and  $C''$  are geometrically linked,  $W$  is smooth and  $C$  is arithmetically Cohen Macaulay, we apply [F3] Prop. 3.1 and we obtain:  $h^0(\mathcal{I}_{Z''|W}(iH + R)) = h^1(\mathcal{I}_{Z|W}(m + w - i)) = d - h_Z(m + w - i)$ . Then note that for every  $k$  we

have  $d - h_Z(k) = d + \Delta h_Z(k+1) - h_Z(k+1) = d + \sum_{r=k+1}^t \Delta h_Z(r) - h_Z(t) = \sum_{r=k+1}^\infty \Delta h_Z(r)$  because for  $t$  big we have  $h_Z(t) = d$ , and that, by Prop. 3.1,  $\Delta h_Z(r) = h(r)$ . By [F3] Cor. 3.10 (if  $X$  is smooth) or Cor. 4.28 (if  $X = S(0, 1, 2)$ ) we have that

$$h^0(\mathcal{I}_{C''|X}(iH + R)) \geq h^0(\mathcal{I}_{Z''|W}(iH + R)). \quad (3.9)$$

From the exact sequence

$$0 \rightarrow \mathcal{I}_{C''|X}((k-1)H + R) \rightarrow \mathcal{I}_{C''|X}(kH + R) \rightarrow \mathcal{I}_{Z''|W}(kH + R) \rightarrow 0$$

we obtain that if for  $k = i-1$  we have  $h^0(\mathcal{I}_{Z''|W}(kH + R)) = 0$ , then  $h^0(\mathcal{I}_{C''|X}(kH + R)) = 0$ . In this hypothesis for  $k = i$  we have an injection  $H^0(\mathcal{I}_{C''|X}(iH + R)) \hookrightarrow H^0(\mathcal{I}_{Z''|W}(iH + R))$  and therefore by (3.9)  $h^0(\mathcal{I}_{C''|X}(iH + R)) = h^0(\mathcal{I}_{Z''|W}(iH + R))$ .  $\square$

Using classical linkage techniques is easy to prove ([F2] Lemma 4.7) the following Lemma:

**Lemma 3.10** *Let  $X = S(0, 0, 3)$ , let  $A$  as in Def. 3.6 and let  $Z'''$  as in Def. 3.7. Then for  $i < w$*

$$h^0(\mathcal{I}_{Z'''|A}(i+1)) = h^1(\mathcal{I}_{Z|X}(m+w-i)).$$

Using Lemma 3.10 and [F3] Th. 4.20 we prove the following

**Lemma 3.11** *If  $X = S(0, 0, 3)$ , then for  $i \leq w$ :*

$$h^0(\mathcal{I}_{C''|X}(iH + R)) \geq h^1(\mathcal{I}_{Z|W}(m+w-i)) = \sum_{r=m+w-i+1}^{\infty} h(r).$$

Moreover if  $i < w$ ,  $h^0(\mathcal{I}_{Z'''|A}(i)) = 0$  and  $h^0(\mathcal{I}_{Z'''|A}(i+1)) = h > 0$ , then  $h^0(\mathcal{I}_{C''|X}((i-1)H + R)) \geq h^0(\mathcal{I}_{C'''|A}(i)) = 0$  and  $h^0(\mathcal{I}_{C''|X}(iH + R)) \geq h^0(\mathcal{I}_{C'''|A}(i+1)) = h$ .

The next result is a formula which relates the arithmetic genera of the curves  $C$ ,  $C''$  and  $Y = X \cap F \cap G$ .

**Lemma 3.12** *Let  $X$  be smooth or  $X = S(0, 1, 2)$ . Let  $C$ ,  $C''$  and  $Y = X \cap F \cap G$  as usual. Then we have the following relation:*

$$p_a(C'') = p_a(C) - p_a(Y) + (m+w-1) \cdot \deg C'' + \deg(R \cap C'') + 1 \quad (3.13)$$

**Proof.** [F3] Prop. 3.11 in the case  $X = S(1, 1, 1)$  and [F3] Th. 4.30 if  $X = S(0, 1, 2)$ .  $\square$

## 4 The classification

At this point we are able to prove the main Theorem. The techniques that we use to prove Theorem 1.1 are the same for the three cases:  $X = S(1, 1, 1)$ ,  $X = S(0, 1, 2)$  and  $X = S(0, 0, 3)$ ; therefore we don't want to give a proof for all cases. On the other side the analysis is slightly different case by case, therefore, to be impartial, we will give the proof of Th. 1.1 2) in case  $X = S(0, 1, 2)$ , of Th. 1.1 3) in case  $X = S(1, 1, 1)$  and of Th. 1.1 4) in case  $X = S(0, 0, 3)$ . For a complete proof the reader may consult [F1]. We will give a more precise description of such curves  $C'$  in the next propositions. The reader may go back to Remark where we have described planes and surfaces of degree 2 or 3 contained in  $X$ , and to the previous section where we have introduced some notation.

**Proof.**[of the Theorem 1.1]

1. Let  $k = 3$ . This happens if and only if  $\epsilon = s - 1$ . It follows  $\deg C' = s - \epsilon - 1 = 0$  and we are done.
2. Let  $k = 2$  and let  $X = S(0, 1, 2)$ . We treat separately the cases  $v = 0, 1, 2$ .
  - Let  $v = 2$ , i.e.  $S \sim (w+1)H$ . Then  $e = 1$  and we have  $\epsilon + 1 = 2(w+1) + \delta$  with  $0 \leq \delta < w + 1$ . By Lemma 3.8 we compute

$$h^0(\mathcal{I}_{C'|X}(R)) = 1.$$

Hence  $C'$  is contained in a plane  $\pi \sim R$  and has degree  $w + 1 \geq \deg C' = 3w + 2 - \epsilon \geq 1$ .

- Let  $v = 1$ , i.e.  $S + p_1 \sim (w + 1)H$ . If  $e = 0$  (i.e. if  $\epsilon < 3w$ ) we have  $\epsilon = 2w + \delta$  with  $0 \leq \delta < w$ . By Lemma 3.8 we compute  $h^0(\mathcal{I}_{C'|X}(R)) = 0$  and  $h^0(\mathcal{I}_{C'|X}(H + R)) = 3$ . Since  $(H + R) \cdot p_1 \cdot H = 1$  and  $\deg C_1 = m + 1 > 1$ , then all the surfaces  $\sim H + R$  containing  $C'$  split in the plane  $p_1 \supset C_1$  and in surfaces  $\sim H - R$  containing  $C'$ . Therefore

$$h^0(\mathcal{I}_{C'|X}(H)) = 3,$$

i.e.  $C'$  is contained in a plane  $\pi$  and has degree  $w + 1 \geq \deg C' = 3w + 1 - \epsilon > 1$ . When  $v = 1$  we have  $e = 1$  only if  $\epsilon = 3w$ . In this case  $\deg C' = 1$ , i.e.  $C'$  is a line.

- Let  $v = 0$ , then  $e = 0$  and we have  $\epsilon = 2w + \delta$  with  $0 \leq \delta < w$ . By Lemma 3.8 we compute  $h^0(\mathcal{I}_{C'|X}(R)) = 0$  and  $h^0(\mathcal{I}_{C'|X}(H + R)) = 2$ . In case  $S + p_1 + p_2 \sim (w + 1)H$ , since  $(H + R) \cdot p_1 \cdot H = (H + R) \cdot p_2 \cdot H = 1$  and  $\deg C_1 = \deg C_2 = m + 1 > 1$ , we find that

$$h^0(\mathcal{I}_{C'|X}(H - R)) = 2.$$

If  $\delta < w - 1$ , then  $\deg C' = 3w - \epsilon = w - \delta > 1$ ; since  $(H - R) \cdot (H - R) \cdot H = 1$ , then the linear system  $|\mathcal{I}_{C'|X}(H - R)|$  has a fixed part which is necessarily the plane  $p \sim H - 2R$ . If  $\delta = w - 1$ , then  $C'$  is a line. Therefore  $C'$  is contained in the plane  $p \sim H - 2R$  or it is a line.

In case  $S+q \sim (w+1)H$ , since  $(H+R) \cdot q \cdot H = 3$  and  $\deg C_q = m+1 > 3$ , we find that

$$h^0(\mathcal{I}_{C'|X}(2R)) = 2.$$

Therefore  $C'$  is contained in a plane  $\pi \sim R$ , which is the fixed part of  $|\mathcal{I}_{C'|X}(2R)|$ .

3. Let  $k = 1$  and let  $X$  be smooth.

• Let  $v = 2$  and  $e = 0$ ; we have  $\epsilon = w + \delta$  with  $0 \leq \delta < w$ . By Lemma 3.8 we compute  $h^0(\mathcal{I}_{C'|X}(R)) = 0$  and

$$h^0(\mathcal{I}_{C'|X}(H+R)) = 3.$$

Since  $\deg C' = s - \epsilon - 1 \geq 5$  and  $(H+R) \cdot (H+R) \cdot H = 5$  we deduce that the linear system  $|\mathcal{I}_{C'|X}(H+R)|$  has a fixed part which has degree less or equal than 2 as one can easily verify (if we suppose, for example, that the fixed part is  $L \sim H$ , then  $h^0(\mathcal{O}_X(H+R-L)) = h^0(\mathcal{O}_X(R)) = 2$ , and we have a contradiction since  $h^0(\mathcal{I}_{C'|X}(H+R-L)) = 3$ ). Therefore the fixed part can be of the following types:

- (a)  $\pi \sim R$ . In this case  $C'$  is the union of a plane curve  $C'_1$  on  $\pi$  and of a curve  $C'_2$  contained in the base locus of a net of hyperplane sections, i.e. in a plane  $\sigma$ . If  $\sigma \sim R$ , then the fixed part of  $|\mathcal{I}_{C'|X}(H+R)|$  is  $\sim 2R$  and we are in the next case (b). The other possibility is that  $\sigma$  does not belong to  $X$ . Since  $\deg C' \geq w+3$  and  $\pi \cdot S \cdot H = w+1$  this is possible only when  $\deg C' = w+3$ , i.e.  $\delta = w-1$  and  $C'_2$  is a curve of type  $(1,1)$  on a quadric surface  $\sim H-R$ . In this case  $C'$  is contained in the surface of degree two  $\pi \cup \sigma$ .
- (b)  $Y \sim 2R$ . Then  $|\mathcal{I}_{C'|X}(H+R)| = Y + |\mathcal{O}_X(H-R)|$  (in fact  $h^0(\mathcal{O}_X(H-R)) = 3$ ). Since  $|\mathcal{O}_X(H-R)|$  is free,  $C'$  is contained in the surface of degree two  $Y \sim 2R$ .
- (c)  $Q \sim H-R$ , i.e.  $|\mathcal{I}_{C'|X}(H+R)| = Q + |\mathcal{O}_X(2R)|$ . Since  $|\mathcal{O}_X(2R)|$  is free  $C'$  is contained in the smooth quadric surface  $Q$ .

When  $v = 2$ , then  $e = 1$  only if  $\epsilon = 2w$ , i.e.  $\deg C' = w+2$ . In this case we write  $\epsilon + 1 = w + 1 + \delta$  with  $\delta = w$ . By Lemma 3.8 and we compute  $h^0(\mathcal{I}_{C'|X}(R)) = 0$  and

$$h^0(\mathcal{I}_{C'|X}(H+R)) = 4.$$

Since  $\deg C' = w+2 \geq 4$  and  $(H+R) \cdot (H+R) \cdot H = 5$  we deduce that the linear system  $|\mathcal{I}_{C'|X}(H+R)|$  has a fixed part which is, as one can easily verify,  $\pi \sim R$ . In this case  $C'$  is the union of a plane curve of degree  $w+1$  on  $\pi$  and of a line.

• Let  $v = 1$ . Then  $e = 0$  and we have  $\epsilon = w + \delta$  with  $0 \leq \delta < w$ . By Lemma 3.8 we compute  $h^0(\mathcal{I}_{C''|X}(R)) = 0$  and  $h^0(\mathcal{I}_{C''|X}(H+R)) = 2$ .

With the same computation we have done previously (case  $k = 2$  and  $v = 1$ ) one can deduce that

$$h^0(\mathcal{I}_{C'|X}(H)) = 2.$$

Since  $\deg C' > w + 1 \geq 3$  and  $H^3 = 3$  the linear pencil  $|\mathcal{I}_{C'|X}(H)|$  should have a fixed part, which can be of the following types:

- (a)  $\pi \sim R$ . In this case  $C'$  is the union of a plane curve of degree  $w + 1 = \pi \cdot S \cdot H$  on  $\pi$  and of a line, which is the base locus of a pencil of quadric surfaces  $\sim H - R$ . This is possible only when  $\deg C' = w + 2$ , i.e.  $\delta = w - 1$ .
- (b)  $Q \sim H - R$ , i.e.  $|\mathcal{I}_{C'|X}(H)| = Q + |\mathcal{O}_X(R)|$ . Since  $|\mathcal{O}_X(R)|$  is free  $C'$  is contained in the smooth quadric surface  $Q$ .

The fixed part of  $|\mathcal{I}_{C'|X}(H)|$  cannot be  $Y \sim 2R$  since in this case we would have  $|\mathcal{I}_{C'|X}(H)| = Y + |\mathcal{O}_X(H - 2R)|$ , while  $h^0(\mathcal{O}_X(H - 2R)) = 0$ .

• Let  $v = 0$ , then  $e = 0$  and we have  $\epsilon = w + \delta$  with  $0 \leq \delta < w$ . By Lemma 3.8 we compute  $h^0(\mathcal{I}_{C''|X}(R)) = 0$  and  $h^0(\mathcal{I}_{C''|X}(H + R)) = 1$ . In case  $S + p_1 + p_2 \sim (w + 1)$ , one easily deduces that

$$h^0(\mathcal{I}_{C'|X}(H - R)) = 1,$$

i.e.  $C'$  is contained in a smooth quadric surface  $Q \sim H - R$ . In case  $S + q \sim (w + 1)H$ , one find that

$$h^0(\mathcal{I}_{C'|X}(2R)) = 1,$$

therefore  $C'$  is contained in a reducible surface of degree two  $Y \sim 2R$ .

4. Let  $k = 0$  and let  $X = S(0, 0, 3)$ . Then  $e = 0$  and we write  $\epsilon = \delta$  with  $0 \leq \delta < w$ .
  - Let  $v = 2$ . By Lemma 3.11 we compute  $h^0(\mathcal{I}_{C'|X}(R)) \geq h^0(\mathcal{I}_{C'''|A}(1)) = 0$  and

$$h^0(\mathcal{I}_{C'|X}(4R)) \geq h^0(\mathcal{I}_{C'''|A}(2)) = 2.$$

Since  $\deg C' \geq 2w + 3 \geq 7$  and  $4R \cdot 4R \cdot H = 6$  by (2.5), we deduce that the linear system  $|\mathcal{I}_{C'|X}(4R)|$  has a fixed part.

We exclude that the fixed part is  $\pi \sim R$ . Indeed in this case  $C'$  would be the union of a curve contained in  $\pi$  of degree at most  $\pi \cdot S \cdot H = w + 1$ , and of a curve contained in the base locus of a pencil of hyperplane sections of degree at most  $H^3 = 3$ . But this is not possible since  $\deg C' \geq w + 5$ .

Also  $Y \sim 2R$  is not possible. In fact in this case  $C'$  would be contained in  $Y$  (since the reduced singular line of  $X$ , which is the base locus of the residual system  $|\mathcal{I}_{C'|X}(4R) - Y| \subset |\mathcal{O}_X(2R)|$ , is contained in  $Y$ ). This can not happen: the quadric hypersurfaces which are union of a

hyperplane containing  $Y$  and of a hyperplane containing  $B$  cut on  $A$  a linear subsystem of  $|\mathcal{I}_{C'''}|_A(2)|$  of projective dimension 2 and this is not possible since  $h^0(\mathcal{I}_{C'''}|_A(2)) = 2$ .

The only possibility is that the fixed part is  $L \sim 3R$ . In this case we claim that  $C'$  is contained in  $L$ . In this case we prove that  $C'$  is contained in a surface of degree 3 which is a hyperplane section of  $X$ .  $C'$  is the union of a curve contained in  $L$  and possibly of the reduced singular line  $l$  of  $X$ , which is the base locus of the residual system  $|\mathcal{I}_{C'}|_X(4R) - Y| = |\mathcal{O}_X(R)|$ . If  $L$  is reducible, then  $l \subset L$  and we are done. If  $L$  is irreducible, i.e.  $L$  does not contain  $l$ , then  $C'$  would contain  $l$  with multiplicity 1, but this is not possible, as the following computation shows. Let  $\tilde{S} \sim (w+1-a)\tilde{H} + 3a\tilde{R}$  ( $0 \leq a \leq w$ ) and  $\tilde{F} \sim (m+1-b)\tilde{H} + 3b\tilde{R}$  ( $0 \leq b \leq m$ ) be the proper transforms of  $S$  and  $F$  in  $\tilde{X}$ . When  $a \geq 1$  and  $b \geq 1$  by (2.6)  $C'$  contains  $l$  with multiplicity  $m(F, S; l) = 3ab$ .

• Let  $v = 1$ . By Lemma 3.11 we compute  $h^0(\mathcal{I}_{C''}|_X(R)) \geq h^0(\mathcal{I}_{C'''}|_A(1)) = 0$  and

$$h^0(\mathcal{I}_{C''}|_X(4R)) \geq h^0(\mathcal{I}_{C'''}|_A(2)) = 1.$$

Since by (2.5)  $4R \cdot p_1 \cdot H = 2$  and  $\deg C_1 = m+1 > 2$ , a surface  $\sim 4R$  which contains  $C''$  splits in the union of  $p_1 \sim R$  and a surface  $\sim 3R \sim H$  which contains  $C'$ . Therefore we have:

$$h^0(\mathcal{I}_{C'}|_X(H)) \geq 1.$$

We claim that we have exactly  $h^0(\mathcal{I}_{C'}|_X(H)) = 1$ . This follows from the fact that a quadric hypersurface that is union of the hyperplane  $H$  which cuts  $p_1 + Y_B$  on  $X$  and of a hyperplane which cuts  $L \in |\mathcal{I}_{C'}|_X(H)|$  is a quadric hypersurface which cuts on  $A$  a divisor in  $|\mathcal{I}_{C'''}|_A(2)|$  and we know that  $h^0(\mathcal{I}_{C'''}|_A(2)) = 1$ .

• Let  $v = 0$ . By Lemma 3.11 we compute  $h^0(\mathcal{I}_{C''}|_W(4R)) \geq h^0(\mathcal{I}_{C'''}|_A(2)) = 0$  and

$$\begin{cases} h^0(\mathcal{I}_{C''}|_W(7R)) \geq h^0(\mathcal{I}_{C'''}|_A(3)) = 4 & \text{if } \epsilon = w-1 \\ h^0(\mathcal{I}_{C''}|_W(7R)) \geq h^0(\mathcal{I}_{C'''}|_A(3)) = 3 & \text{if } \epsilon < w-1 \end{cases}$$

Since by (2.5)  $7R \cdot p_1 \cdot H = 7R \cdot p_2 \cdot H = 3$  and  $\deg C_1 = \deg C_2 = m+1 > 3$  we deduce that

$$\begin{cases} h^0(\mathcal{I}_{C'}|_W(5R)) \geq 4 & \text{if } \epsilon = w-1 \\ h^0(\mathcal{I}_{C'}|_W(5R)) \geq 3 & \text{if } \epsilon < w-1 \end{cases}$$

and using similar arguments as in the previous case one can easily prove that equality holds. We claim that the linear system  $|\mathcal{I}_{C'}|_W(5R)|$  has a fixed part. To prove the claim we need first to analyze when  $C'$  may contain the singular line  $l$  of  $X$  as a component. Let  $\tilde{S} \sim a\tilde{H} + (3w -$

$3a+1)\tilde{R}$  ( $0 < a \leq w$ ) and  $\tilde{F} \sim (m+1-b)\tilde{H} + 3b\tilde{R}$  ( $0 \leq b \leq m$ ) be the proper transforms of  $S$  and  $F$  in  $\tilde{X}$ . When  $b \geq 1$  by (2.6)  $C'$  contains  $l$  with multiplicity  $m(F, S; l) = 3b(w-a) + b \geq b$ . On the other hand  $C'$  is contained in the scheme  $S \cap D$  for some  $D \in |\mathcal{I}_{C'|W}(5R)|$ ; since  $S \cap D$  contains  $l$  with multiplicity  $m(D, S; l) = 2(w-a) + 1$  if  $\tilde{D} \sim \tilde{H} + 2\tilde{R}$  or  $m(D, S; l) = 5(w-a) + 2$  if  $\tilde{D} \sim 5\tilde{R}$ , then  $m(F, S; l)$  should be less or equal then these values. Therefore when  $b \geq 1$ , since  $3b(w-a) + b > 2(w-a) + 1$ , we exclude the possibility  $\tilde{D} \sim \tilde{H} + 2\tilde{R}$ ; when  $b \geq 2$ , since  $3b(w-a) + b > 5(w-a) + 2$ , we exclude the possibility  $\tilde{D} \sim 5\tilde{R}$ . Hence  $C'$  may contain  $l$  only if  $b = 1$ , i.e.  $\tilde{F} \sim m\tilde{H} + 3\tilde{R}$ , and the divisors in the linear system  $|\mathcal{I}_{C'|X}(5R)|$  are all reducible in the union of five planes. In this case  $C'$  contains  $l$  with multiplicity  $m(F, S; l) = 3(w-a) + 1$ , which has to be less or equal than  $m(F, 5R; l)$ , that is 5 by (2.6). This is possible only if either  $a = w-1$ , or  $a = w$ . When  $a = w-1$  we have  $\tilde{S} \sim (w-1)\tilde{H} + 4\tilde{R}$  and  $C'$  contain  $l$  with multiplicity 4. When  $a = w$  we have  $\tilde{S} \sim w\tilde{H} + \tilde{R}$  and  $C'$  contain  $l$  with multiplicity 1.

Now we are able to prove that  $|\mathcal{I}_{C'|X}(5R)|$  has a fixed part. Let us suppose first that  $C'$  contains  $l$ . In this case if the linear system  $|\mathcal{I}_{C'|X}(5R)|$  has no fixed part, then  $C'$  is supported on the line  $l$ . Our previous computation implies that  $\deg C' \leq 4$ , while we know that  $\deg C' \geq 7$ . Therefore  $|\mathcal{I}_{C'|X}(5R)|$  has a fixed part as claimed. Let us suppose that  $C'$  does not contain  $l$ . If  $|\mathcal{I}_{C'|X}(5R)|$  has no fixed part, then the generic element  $D$  in  $|\mathcal{I}_{C'|X}(5R)|$  is irreducible and has proper transform  $\tilde{D} \sim \tilde{H} + 2\tilde{R}$ , therefore for  $D, D'$  in the linear system we have  $m(D, D'; l) = 1$ . In this case, since by (2.5)  $5R \cdot 5R \cdot H = 8$ , the base locus of a pencil in  $|\mathcal{I}_{C'|X}(5R)|$  not supported on  $l$  has degree 7. Since  $\deg C' \geq 7$  and  $h^0(\mathcal{I}_{C'|X}(5R)) = 3$  we have a contradiction. Therefore  $|\mathcal{I}_{C'|X}(5R)|$  has a fixed part.

We claim first that this fixed part cannot be  $\pi \sim R$ . In this case  $C'$  would be the union of a curve  $C'_1 \subset \pi$  and of a curve  $C'_2$  contained in the base locus of a linear subsystem  $|\alpha| \subset |\mathcal{I}_{C'|X}(4R)|$  of projective dimension 3 if  $\epsilon = w-1$ , 2 if  $\epsilon < w-1$ . We want to prove that  $|\alpha|$  has a fixed part. Let us suppose first that  $C'$  contains  $l$ , with multiplicity 4 (if  $a = w-1$ ) or 1 (if  $a = w$ ) by our previous computation. If  $|\alpha|$  has no fixed part then  $C'_2$  is supported on  $l$ , and since the component of  $C'_1$  disjoint from the line  $l$  has degree equal to  $\tilde{\pi} \cdot \tilde{S} \cdot \tilde{H} = a$ , we should have  $\deg C' = 4 + w - 1 = w + 3$  (if  $a = w-1$ ) or  $\deg C' = 1 + w$  (if  $a = w$ ), but this is not possible since  $\deg C' = s - \epsilon - 1 = 3w - \epsilon > 2w \geq w + 3$ . Therefore we have a contradiction and  $|\alpha|$  has a fixed part. With similar arguments it is easy to prove that  $|\alpha|$  has a fixed part if we suppose that  $C'$  does not contain  $l$ . The fixed part of  $|\mathcal{I}_{C'|X}(5R)|$  can be of the following types:

- (a)  $Y \sim 2R$ . In this case  $C' = C'_1 \cup C'_2$ , where  $C'_1 \subset Y$  and  $C'_2$  is contained in the base locus of a linear system  $|\beta| \subset |\mathcal{O}_X(H)|$  of projective dimension 3 if  $\epsilon = w-1$ , 2 if  $\epsilon < w-1$ . If  $\epsilon = w-1$ , then  $C'_2$  is a line  $r \subset \pi \sim R$ . Therefore  $C'$  is contained in the

cubic surface  $Y \cup \pi \sim 3R$ . If  $\epsilon < w - 1$ , then  $C'_2 \subset \sigma$  is a plane curve contained in a plane  $\sigma$ . If  $C'$  contains  $l$ , the divisors in  $|\beta|$  are reducible in the union of 3 planes, therefore  $\sigma \sim R$  is a fixed part for  $|\mathcal{I}_{C'|X}(5R)|$  and we are in the next case (b). If  $C'$  does not contain  $l$ , then  $\deg C'_1 \leq 2w$ ; therefore if  $\epsilon \leq w - 3$  we have  $\deg C'_2 \geq 3$  which implies  $\sigma \sim R$ . If  $\epsilon = w - 2$  and  $\deg C'_1 = 2w$  (i.e.  $\tilde{S} \sim w\tilde{H} + \tilde{R}$ ), then  $\deg C'_2 = 2$  and  $\sigma$  may be a plane not contained in  $X$ .  $C'$  is contained in the cubic surface  $Y \cup \sigma$ .

- (b)  $L \sim 3R$ . Here we must put  $\epsilon < w - 1$ , since for  $\epsilon = w - 1$  we have  $h^0(\mathcal{I}_{C'|X}(5R)) = 4$ , while  $h^0(\mathcal{O}_X(5R - L)) = h^0(\mathcal{O}_X(2R)) = 3$ . In this case  $C'$  is contained in a cubic surface  $L \sim 3R$ , hyperplane section of  $X$ .

□

In the next propositions we give a closer description of case 2) in Th. 1.1 when  $X = S(0, 1, 2)$ , of case 3) when  $X$  is smooth and of case 4) when  $X = S(0, 0, 3)$ . In all the other cases one can give a similar description, but we don't intend to go through this.

**Proposition 4.1** *Let  $k = 2$ . If  $X = S(0, 1, 2)$  then we have the following possibilities:*

1. When  $v = 2$  then  $C'$  is a plane curve of degree  $1 \leq \deg C' \leq w + 1$  contained in a plane  $\pi \sim R$ .  $C'$  does not pass through the singular point  $V$  of  $X$ .
2. When  $v = 1$  then  $C'$  is contained in a plane which can be of the following types:
  - (a) if  $\deg C' = w + 1$ , i.e.  $\epsilon = 2w$ , then  $C' \subset \pi \sim R$  and passes through the vertex  $V$  of  $X$ .
  - (b) if  $\deg C' \leq w$ , i.e.  $2w < \epsilon \leq 3w$ , then  $C'$  lies either in  $\pi \sim R$  or in  $p \sim H - 2R$  and may pass or not through  $V$ .
  - (c) if  $\deg C' = 2$ , i.e. if  $\epsilon = 3w - 1$  there is the further possibility that  $C'$  is a conic lying on a plane  $\sigma$  which does not belong to the scroll. In this case  $C'$  is either the union of two lines passing through  $V$ , or it is a hyperplane section of a quadric cone  $\sim H - R$ .
3. When  $v = 0$ ,  $S \sim (w + 1)H - 2R$  and  $1 \leq \deg C' \leq w - 1$  (i.e.  $2w < \epsilon \leq 3w - 1$ ) then  $C'$  is contained in  $p \sim H - 2R$ .  $C'$  may pass or not through the vertex of  $X$ . When  $\deg C' = 1$ , i.e. if  $\epsilon = 3w - 1$ , there is the further possibility that  $C'$  is a line contained in a plane  $\pi \sim R$  and passing through  $V$ .
4. When  $v = 0$  and  $S \sim wH + R$ , then  $C'$  is a plane curve of degree  $1 \leq \deg C' \leq w$  contained in  $\pi \sim R$ .  $C'$  may pass or not through the vertex of  $X$ .



**Proof.**

1. This is the case  $k = v = 2$  in the proof of the Theorem 1.1. That  $C'$  does not pass through the point  $V$  follows by genus formula (3.13): in this case in fact  $\deg(R \cap C') \geq 1$  and this would imply that  $p_a(C)$  is not maximal.
2. Looking at the case  $k = 2$  and  $v = 1$  in the proof of Theorem 1.1 we know that  $C'$  is contained in a plane  $\pi$  which is the base locus of a net of hyperplanes. If  $\deg C' \geq 3$  this plane must be contained in the scroll, therefore is either  $\pi \sim R$  or  $\pi = p \sim H - 2R$ .
  - (a) When  $\deg C' = w + 1$ , since  $S \cdot p \cdot H = w$  we must have  $\pi \sim R$ . Since  $S \sim (w + 1)H - R$ , the hypersurface  $G$  of degree  $w + 1$  which cut  $S$  on  $X$  must pass through the point  $V$ ; therefore the curve  $C'$  must contain  $V$ .
  - (b) When  $\deg C' < w + 1$ , by genus formula (3.13) one verifies that  $\pi$  can be both  $\pi \sim R$  or  $\pi = p \sim H - 2R$  and that  $C'$  may pass or not through the vertex  $V$ .
  - (c) If  $\deg C' = 2$  the plane that contains  $C'$  may not belong to the scroll. Therefore, as we have explained in Remark , either  $C'$  is the union of two lines meeting at  $V$ , or it is a hyperplane section of a quadric cone  $\sim H - R$
3. Looking at the proof of Theorem 1.1 when  $k = 2$  and  $S \sim (w + 1)H - 2R$  we know that if  $\deg C' > 1$  then  $C'$  lies on the plane  $p \sim H - 2R$ . In this case, since  $S \cdot p \cdot H = w - 1$ , we must exclude the case  $\deg C' = w$ . If  $\deg C' = 1$  then  $C'$  is a line which can also be the base locus of a pencil of quadric cones  $\sim H - R$ , therefore in this case it lies in a plane  $\pi \sim R$  and passes through  $V$ .
4. This is proved in Theorem 1.1. Since  $\deg C' = 3w - \epsilon$  and  $2w \leq \epsilon \leq 3w - 1$ , we have that  $1 \leq \deg C' \leq w$ .

□

**Proposition 4.2** *If  $k = 1$ , then  $C'$  is contained in a surface of degree two. When  $X = S(1, 1, 1)$  we have the following possibilities:*

1. *if  $S \approx wH + R$  and  $\epsilon = w, w + 1$  the surface may be a smooth quadric surface  $Q \sim H - R$ . In this case if  $\epsilon = w$  then  $C'$  is a curve of kind  $(w - 1 + v, w + 1)$  on  $Q$ ; if  $\epsilon = w + 1$  then  $C'$  is of kind  $(w - 1 + v, w)$  on  $Q$ .*
2. *If  $v = 2$  or  $v = 0$  and  $S \sim wH + R$  the surface may be reducible in the union of two disjoint planes  $\pi_1 \sim R$  and  $\pi_2 \sim R$ . In this case  $C' = C'_1 \cup C'_2$ , where  $C'_1$  is a curve of degree  $2w + 1 - \epsilon$  if  $v = 2$  (resp. of degree  $2w - \epsilon$  if  $S \sim wH + R$ ) on  $\pi_1$  and  $C'_2$  is a curve of degree  $w + 1$  (resp.  $w$ ) on  $\pi_2$ .*

3. When a)  $v = 2$  and  $\epsilon = 2w - 1$  or b)  $v = 2$  and  $\epsilon = w$  or c)  $v = 1$  and  $\epsilon = 2w - 1$ , the surface may be the union of a plane  $\pi \sim R$  and a plane  $\sigma$  not contained in the scroll  $X$ . In this cases  $C' = C'_1 \cup C'_2$ , where  $C' \subset \pi$  has degree  $w + 1$  and  $C'_2 \subset \sigma$  is a) a conic of kind  $(1, 1)$  or b), c) a line of kind  $(0, 1)$ .  $C'_1$  and  $C'_2$  intersect each other in a point.
4. if  $v = 1$  and  $\epsilon \neq w, w + 1, 2w - 1$  or  $S \sim (w + 1)H - 2R$  and  $\epsilon \neq w, w + 1$  there are no curves of maximal genus  $G(5, d, s)$  on  $X = S(0, 0, 1)$ .

**Proof.**

1. Let  $S$  and  $\Gamma$  be as usual. Looking at the cases  $v = 2$  part c),  $v = 1$  part b) and  $S \sim (w + 1)H - 2R$  in the proof of Theorem 1.1 we know that  $C'$  may be contained in a smooth quadric surface  $Q \sim H - R$ . Hence the hypersurfaces in  $\mathbb{P}^4$  of degree  $m + 1$  through  $Z$  cut on  $\Gamma$  a linear series  $|\gamma|$  of degree  $\deg Z' = s - \epsilon - 1$ , whose divisors are contained in conics  $\sim H - R$ , and therefore in hyperplane sections. Since  $\Gamma$  is arithmetically Cohen Macaulay, it follows that  $|\gamma|$  is cut out on  $\Gamma$  by the linear system of hyperplanes through  $H \cdot \Gamma - \deg Z' = \epsilon + 1$  points. The projective dimension of  $|\gamma|$  is equal to  $h_\Gamma(m + 1) - h_Z(m + 1)$ . If  $\epsilon \geq w + 1$  we have  $\dim |\gamma| = 1$ , therefore the  $\epsilon + 1$  points should span a  $\mathbb{P}^2$ . If  $\epsilon = w$  we have  $\dim |\gamma| = 2$ , hence they lie on a line  $l \sim R$  on  $W$ . If  $\epsilon \geq w + 1$  the movable part of  $|\gamma|$  is cut out by the pencil of conics  $\sim H - R$  through a fixed point (since  $h^0(\mathcal{O}_W(H - R)) = 3$ ); this implies that the remaining  $\epsilon$  points lie on a line  $\sim R$ . Therefore  $\epsilon + 1 = \Gamma \cdot R + 1 = w + 2$ , i.e.  $\epsilon = w + 1$ . In this case  $Z'$  is a set of  $\Gamma \cdot (H - R) - 1 = 2w + v - 1$  points on the conic  $\sim H - R$ , hence  $C'$  is linked to a line of kind  $(0, 1)$  on a smooth quadric surface  $Q \sim H - R$  by the intersection with  $S$ , i.e.  $C'$  is a curve of type  $(w - 1 + v, w)$ . If  $\epsilon = w$ , the movable part of  $|\gamma|$  is cut out by the linear system  $|\mathcal{O}_W(H - R)|$  and  $Z'$  is a set of  $\Gamma \cdot (H - R) = 2w + v$  points on a conic  $\sim H - R$ . Therefore  $C'$  is the intersection of  $S$  with a smooth quadric surface  $Q$ , hence is a curve of type  $(w - 1 + v, w + 1)$ .
2. We omit the proof in this case, since it goes on similarly as the previous one.
3. This third possibility follows directly from the proof of Theorem 1.1.
4. If  $v = 1$  from the proof of Theorem 1.1 we know that  $C'$  lies on a smooth quadric surface  $\sim H - R$  or, if  $\epsilon = 2w - 1$ , in the surface  $\pi \cup \sigma$  of part 3) in the statement. Therefore by part 1) of this proposition we deduce that for  $\epsilon \neq w, w + 1, 2w - 1$  the curve  $C'$  does not exist. In a similar way one deduces that  $C'$  does not exist when  $S \sim (w + 1)H - 2R$  and  $\epsilon \neq w, w + 1$ .

□

The proof of the next proposition is omitted since the argument is similar to the previous one. Also the explicit computation for the multiplicity of the line  $l$  as a component of the linked curve  $C'$  is similar to the one appearing in the proof of the Theorem part 4).

**Proposition 4.3** *If  $k = 0$ , then  $C'$  is contained in a surface of degree 3. When  $X = S(0, 0, 3)$  we have the following possibilities:*

1. *The surface is a hyperplane section  $L$  of  $X$ . More precisely:*

- (a) *if  $\epsilon = 0$ , then  $C'$  is linked to a line by the intersection  $S \cap L$ ;*
- (b) *if  $\epsilon = 1$ , then  $C'$  is linked to a conic by the intersection  $S \cap L$ ;*
- (c) *if  $\epsilon > 1$ , then  $L$  split in the union of three planes  $\pi_i \sim R$  ( $i = 1, 2, 3$ ). In this case, when  $v = 2$  the proper transform of  $S$  is  $\tilde{S} \sim (w + 1 - a)\tilde{H} + 3a\tilde{R}$  with  $0 \leq a \leq w - \epsilon$ .  $C'$  is the union of the line  $l$  counted with multiplicity  $3a$ , of a curve  $C'_1 \subset \pi_1$  of degree  $w + 1 - a$ , of a curve  $C'_2 \subset \pi_2$  of the same degree, and of a curve  $C'_3 \subset \pi_3$  of degree  $w - \epsilon - a$ .  $C'_1$  and  $C'_2$  intersect each other in  $w + 1 - a$  points on  $l$ .  $C'_1$  and  $C'_2$  both intersect  $C'_3$  in  $w - \epsilon - a$  points on  $l$ .*

*When  $v = 1$  the proper transform of  $S$  is  $\tilde{S} \sim a\tilde{H} + (3w - 3a + 2)\tilde{R}$  with  $\epsilon + 1 \leq a \leq w$ .  $C'$  is the union of the line  $l$  counted with multiplicity  $3(w - a) + 2$ , of a curve  $C'_1 \subset \pi_1$  of degree  $a$ , of a curve  $C'_2 \subset \pi_2$  of the same degree, and of a curve  $C'_3 \subset \pi_3$  of degree  $a - \epsilon - 1$ .  $C'_1$  and  $C'_2$  intersect each other in  $w + 1 - a$  points on  $l$ .  $C'_1$  and  $C'_2$  both intersect  $C'_3$  in  $w - \epsilon - a$  points on  $l$ .*

*When  $v = 0$  the proper transform of  $S$  is  $\tilde{S} \sim w\tilde{H} + \tilde{R}$ . If  $\epsilon = w - 1$ , then  $C'$  does not contain the line  $l$  and it is the union of a curve  $C'_1 \subset \pi_1$  of degree  $w$ , of a curve  $C'_2 \subset \pi_2$  of the same degree, and of a curve  $C'_3 \subset \pi_3$  of degree  $w - \epsilon$ .  $C'_1$  and  $C'_2$  intersect each other in  $w$  points on  $l$ .  $C'_1$  and  $C'_2$  both intersect  $C'_3$  in  $w - \epsilon$  points on  $l$ . If  $\epsilon < w - 1$ , then  $C'$  may contain the line  $l$  with multiplicity 1. In this case  $C'$  is the union of  $l$ ,  $C'_1$ , of  $C'_2$ , and of a curve  $C'_3 \subset \pi_3$  of degree  $w - \epsilon - 1$ .  $C'_1$  and  $C'_2$  intersect each other in  $w$  points on  $l$ .  $C'_1$  and  $C'_2$  both intersect  $C'_3$  in  $w - \epsilon - 1$  points on  $l$ .*

- 2. *If  $v = 0$  and  $\epsilon = w - 2$  the surface may split in the union of a plane  $\pi_1 \sim R$ , of a plane  $\pi_2 \sim R$  and of a plane  $\sigma$  which is not contained in the scroll.  $C'$  does not contain the line  $l$  as a component and is the union of a curve  $C'_1 \subset \pi_1$  of degree  $w$ , of a curve  $C'_2 \subset \pi_2$  of the same degree, and of the lines  $r_1$  and  $r_2$  of intersection between  $\sigma$  and  $X$ .  $C'_1$  and  $C'_2$  intersect each other in  $w$  points on  $l$ .  $r_1$  and  $r_2$  intersect  $C'_1$  and  $C'_2$  in their common point on  $l$ .*

## 5 The existence

Lastly we need an effective construction for curves of degree  $d$ , genus  $G(d, 5, s)$  in  $\mathbb{P}^5$ , not lying on surfaces of degree  $< s$ . It should be noted that in case  $k = v = 1$  it is not possible to construct curves of maximal genus on a smooth rational normal 3-fold (see Prop. 4.2 case 4)). In this case the construction is possible only on a rational normal 3-fold whose vertex is a point. Before the

construction we state the following, easy to prove, result (see [Ro] Lemma 1 pg. 133), that we will use.

**Lemma 5.1** *Let  $X$  be a smooth 3-fold. Let  $\Sigma$  be a linear system of surfaces of  $X$  and let  $\gamma$  be a curve contained in the base locus of  $\Sigma$ . Suppose that the generic surface of  $\Sigma$  is smooth at the generic point of  $\gamma$  and that it has at least a singular point which is variable in  $\gamma$ . Then all the surfaces of  $\Sigma$  are tangent along  $\gamma$ .*

**Example 5.2** *For all  $d > \frac{2s}{3} \prod_{i=1}^3 (4!)^{\frac{1}{4-i}}$  and  $s \geq 4$ , there is a smooth curve  $C \subset \mathbb{P}^5$  of degree  $d$ , genus  $G(d, 5, s)$  contained on an irreducible surface  $S$  of degree  $s$ .*

- Let  $k = 3$ . In this case take  $S$  to be the complete intersection of a smooth rational normal 3-fold  $X \subset \mathbb{P}^5$  and a general hypersurface  $G$  of degree  $w + 1$  in  $\mathbb{P}^5$ . The complete intersection of  $S$  with a general hypersurface  $F$  of degree  $m + 1$  gives the required curve  $C$ .

- Let  $k = 2$  and  $v = 2$ . In this case we have  $\epsilon + 1 = 2(w + 1) + \delta$  with  $0 \leq \delta \leq w$ . Let  $\pi$  be a plane contained in a smooth rational normal 3-fold  $X$  in  $\mathbb{P}^5$  and let  $D \subset \pi$  be a smooth plane curve of degree  $w + 1 > \deg D = \epsilon - 2w - 1 \geq 0$  (possibly  $D = \emptyset$ ). Let us consider the linear system  $|\mathcal{I}_{D|X}(w + 1)|$ , which is not empty since contains the linear subsystem  $L + |\mathcal{O}_X(w)|$ , where  $L$  is a hyperplane section of  $X$  containing the plane  $\pi$ . This shows also that  $|\mathcal{I}_{D|X}(w + 1)|$  is not composed with a pencil because in this case every element in the system would be a sum of algebraically equivalent divisors, but, for example, the elements in  $L + |\mathcal{O}_X(w)|$  are obviously not of this type. Since  $\deg D < w + 1$  the linear system  $|\mathcal{I}_{D|\pi}(w + 1)|$  is not empty and its base locus is the curve  $D$ ; this shows that the base locus of  $|\mathcal{I}_{D|X}(w + 1)|$  is exactly the curve  $D$ , because  $|\mathcal{O}_X(w)|$  is base points free. Therefore by Bertini's Theorem the general divisor in  $|\mathcal{I}_{D|X}(w + 1)|$  is an irreducible surface  $S$  of degree  $s$ , which is smooth outside  $D$ . We claim that  $S$  is in fact smooth at every point  $p$  of  $D$ . To see this, by Lemma 5.1, it is enough to prove that, for every point  $p \in D$ , there exists a surface in  $|\mathcal{I}_{D|X}(w + 1)|$  which is smooth at  $p$ , and that for a generic point  $q \in D$ , there exist two surfaces in  $|\mathcal{I}_{D|X}(w + 1)|$  with distinct tangent planes at  $q$ . Indeed, for every  $p \in D$  we can always find a surface  $T$  in the linear system  $|\mathcal{O}_X(w)|$  which does not pass through  $p$ , therefore the surface  $L + T$  is smooth at  $p$  with tangent plane  $\pi$ . Moreover a generic surface in the linear system  $|\mathcal{I}_{D|X}(w + 1)|$  which cut  $D$  on  $\pi$  has at  $p$  tangent plane  $T_p \neq \pi$ .

Let  $C' \subset \pi$  be the linked curve to  $D$  by the intersection of  $S$  and  $\pi$ . Let us consider the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ . Since  $\deg C' = s - \epsilon - 1 < m + 1$ , with the same arguments used above it is easy to see that  $|\mathcal{I}_{C'|S}(m + 1)|$  is not empty, is not composed with a pencil and has base locus equal to the curve  $C'$ . Therefore by Bertini's Theorem the generic curve  $C$  in the movable part of this linear system is irreducible and smooth. Moreover  $C$  lies on a smooth surface  $S$  of degree  $s = 3w + 3$  and it has the required numerical characters, as one may

easily verify using in the genus formula (3.13)  $p_a(C') = \frac{1}{2}(3w + 2 - \epsilon - 1)(3w + 2 - \epsilon - 2)$  (computed by Clebsch's formula) and  $\deg(R \cap C') = 0$ .

- Let  $k = 2$  and  $v = 1$ . In this case we have  $\epsilon = 2w + \delta$  with  $0 \leq \delta \leq w$ . Let  $X$  and  $\pi \subset X$  be as in the previous case and let  $p_1 \neq \pi$  be an other plane contained in  $X$ . Let  $D \subset \pi$  be a smooth plane curve of degree  $0 \leq \deg D = \epsilon - 2w < w + 1$  on  $\pi$ . In this case by Bertini's Theorem we can find an irreducible surface  $S \sim (w + 1)H - R$  of degree  $s = 3w + 2$  in the movable part of the linear system  $|\mathcal{I}_{D \cup p_1|X}(w + 1)|$ , smooth outside  $D$ . With the same argument used in the previous case we prove that  $S \cup p_1$  is smooth at every point of  $D$ . Namely, for every  $p \in D$ , a generic surface in the linear system  $L + L_1 + |\mathcal{O}_X(w - 1)|$ , where  $L = \pi + Q$  and  $L_1 = p_1 + Q_1$  are reducible hyperplane sections containing respectively  $\pi$  and  $p_1$  and such that  $p \notin Q \cup Q_1$ , is smooth at  $p$  with tangent plane  $\pi$ , while a surface in the linear system  $L_1 + |\mathcal{I}_{D|X}(w)|$  which cut  $D$  on  $\pi$ , has tangent plane  $T_p \neq \pi$ . Since  $D \cap p_1 = \emptyset$  this implies that  $S$  is smooth.

Let  $C'$  be the linked curve to  $D$  by the intersection  $\pi \cap S$ . By Bertini's Theorem the linked curve  $C$  to  $C'$  by the intersection of  $S$  with a general element  $F_{m+1}$  in the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$  is smooth of degree  $d$ . Moreover it lies on a smooth surface  $S$  of degree  $s = 3w + 2$  and it has the required genus, as one can compute by using formula (3.13), where the curve  $C''$  is the union of  $C'$  and of the curve  $C_1 \subset p_1$  cut on  $p_1$  by  $F_{m+1}$ , and  $\deg(R \cap C'') = 0$ .

- Let  $k = 2$  and  $v = 0$ . In this case we have  $\epsilon = 2w + \delta$  with  $0 \leq \delta < w$ . Let  $X$  and  $\pi \subset X$  as before and let  $q$  be a smooth quadric surface contained in  $X$ , intersecting  $\pi$  along a line  $r$ . Let  $D \subset \pi$  be a smooth plane curve of degree  $0 \leq \deg D = \epsilon - 2w < w$  on  $\pi$  (when  $\deg D = 1$  we suppose that  $D$  does not coincide with the line  $r$ ). In this case by Bertini's Theorem we can find an irreducible surface  $S \sim wH + R$  of degree  $s = 3w + 1$  in the movable part of the linear system  $|\mathcal{I}_{D \cup q|X}(w + 1)|$ , smooth outside  $D$ . As in the previous cases we claim that  $S$  is smooth. Namely, for every  $p \in D$ , we can find in the movable part of our linear system  $|\mathcal{I}_{D \cup q}(w + 1)|$ , a surface which is smooth at  $p$  with tangent plane equal to  $\pi$  (take a surface of the form  $L + \pi' + T$ , with  $L$  as usual,  $\pi' \sim R$  a plane of  $X$  disjoint from  $\pi$  and  $T$  a surface in  $|\mathcal{O}_X(w - 1)H|$  that does not pass through  $p$ ), and a surface with tangent plane  $T_p \neq \pi$  (take a surface of the form  $\pi' + V$ , where  $V$  is a divisor in  $\mathcal{I}_{D|X}(w)$  which cut  $D$  on  $\pi$ ).

Let  $C'$  be the linked curve to  $D$  by the intersection  $\pi \cap S$ ; we have  $\deg C' = 3w - \epsilon$ . The linked curve  $C$  to  $C'$  by the intersection of  $S$  with a general element  $F_{m+1}$  in the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$  is smooth of degree  $d$  and lies on the smooth surface  $S$  of degree  $s = 3w + 1$ . Its genus is maximal as one can compute by using formula (3.13), where the curve  $C''$  is the union of  $C'$  and of the curve  $C_q \subset q$  of type  $(m + 1, m + 1)$  on  $q$ , intersecting each other in  $3w - \epsilon$  points along  $r$ . In this case  $\deg(R \cap C'') = m + 1$ .

- Let  $k = 1$  and  $v = 2$ . In this case we have  $\epsilon = w + \delta$  with  $0 \leq \delta \leq w$ . Let  $\pi_1$  be a plane contained in a smooth rational normal 3-fold  $X$  in  $\mathbb{P}^5$  and let  $D \subset \pi_1$  be a smooth plane curve of degree  $0 \leq \deg D = \epsilon - w < w + 1$  (possibly  $D = \emptyset$ ). In this case by Bertini's Theorem we can find an irreducible

surface  $S$  of degree  $s = 3w + 3$  in the linear system  $|\mathcal{I}_{D|X}(w + 1)|$ , which is also smooth (by using exactly the same argument used in case  $k = v = 2$ ). Let  $\pi_2$  be an other plane contained in  $X$  but not contained in  $S$  and let  $C'$  be the linked curve to  $D$  by the intersection  $S \cap (\pi_1 \cup \pi_2)$ . Therefore  $C'$  is the disjoint union of a plane curve of degree  $2w + 1 - \epsilon$  on  $\pi_1$  and of a plane curve of degree  $w + 1$  on  $\pi_2$ . Let us consider the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ , whose base locus is the curve  $C'$ . Therefore by Bertini's Theorem the generic curve  $C$  in the movable part of this linear system is smooth, of degree  $d$ , lies on a smooth surface  $S$  of degree  $s = 3w + 3$  and it has the required genus as one may easily verify using genus formula (3.13).

- Let  $k = 1$  and  $v = 1$ . In this case  $\epsilon = w + \delta$  with  $0 \leq \delta < w$ . Let  $X$  be a rational normal 3-fold singular at a point  $V$ , i.e.  $X = S(0, 1, 2)$ . Let  $p_1 \sim R$  be a plane in the ruling of  $X$  intersecting the plane  $p \sim H - 2R$  along a line  $r_1$ . Let  $D \subset p$  be a smooth plane curve of degree  $0 \leq \deg D = \epsilon - w < w$  contained in the plane  $p$  and not passing through the vertex  $V$  of  $X$  (in particular when  $\deg D = 1$ , the curve  $D$  can not be the line  $r_1$ ). By Bertini's Theorem we can find an irreducible surface  $S \sim (w + 1)H - R$  of degree  $s = 3w + 2$  in the movable part of the linear system  $|\mathcal{I}_{D \cup p_1|X}(w + 1)|$  smooth outside  $D$  and  $V$ . With the same argument used in case  $k = 2$  and  $v = 1$  one can say that  $S$  is smooth outside the vertex  $V$  of  $X$ . Let  $\pi \sim R$  be an other plane in the ruling of  $X$  and not contained in  $S$ ; let  $C'$  be the linked curve to  $D$  by the intersection  $S \cap (p \cup \pi)$ . Therefore  $C'$  is the union of a plane curve  $C'_p$  of degree  $2w - \epsilon$  on  $p$  which does not pass through  $V$ , and of a plane curve  $C'_\pi$  of degree  $w + 1$  on  $\pi$  passing once through  $V$ , that meet each other in  $2w - \epsilon$  points on the line  $r$  of intersection between  $p$  and  $\pi$ . Let us consider the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ , whose base locus is the curve  $C'$ . Therefore by Bertini's Theorem the generic curve  $C$  in the movable part of this linear system is irreducible, of degree  $d$ , lies on an irreducible surface  $S$  of degree  $s = 3w + 2$  and it is smooth outside  $V$ . But since  $V$  does impose just one condition to the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ , namely the one imposed by  $C'_\pi$ , it follows that  $C$  does not pass through  $V$ , hence  $C$  is smooth. The genus of  $C$  is maximal and can be computed by using formula (3.13), where the curve  $C''$  is the union of  $C'$  and of a plane curve  $C_1 \subset p_1$  of degree  $m + 1$  intersecting along  $2w - \epsilon$  points on the line  $r_1 = \pi \cap p_1$ , and  $\deg(R \cap C'') = 2w - \epsilon + 1$ .

- Let  $k = 1$  and  $v = 0$ . In this case  $\epsilon = w + \delta$  with  $0 \leq \delta < w$ . Let  $X$  be again a smooth rational normal 3-fold in  $\mathbb{P}^5$ . Let  $q \sim H - R$  be a smooth quadric surface contained in  $X$  and let  $\pi_1 \sim R$  be a plane of  $X$  intersecting  $q$  along a line  $r_1$ . Let  $D \subset \pi_1$  be a smooth plane curve of degree  $0 \leq \deg D = \epsilon - w < w$  contained in  $\pi_1$  (when  $\deg D = 1$  we suppose that  $D$  does not coincide with the line  $r_1$ ). With the same argument used in case  $k = 2$  and  $v = 0$  we find an irreducible smooth surface  $S \sim wH + R$  of degree  $s = 3w + 2$  in the movable part of the linear system  $|\mathcal{I}_{D \cup q|X}(w + 1)|$ . Let  $\pi_2 \sim R$  be an other plane of  $X$  not contained in  $S$ ; let  $C'$  be the linked curve to  $D$  by the intersection  $S \cap (\pi_1 \cup \pi_2)$ . Therefore  $C'$  is the disjoint union of a plane curve of degree  $2w - \epsilon$  on  $\pi_1$  and of a plane curve of degree  $w$  on  $\pi_2$ . Let us consider the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ ,

whose base locus is the curve  $C'$ . Therefore by Bertini's Theorem the generic curve  $C$  in the movable part of this linear system is smooth, of degree  $d$  and lies on a smooth surface  $S$  of degree  $s = 3w + 1$ . The genus of  $C$  is maximal and can be computed using formula (3.13), where the curve  $C''$  is the union of  $C'$  and of a curve  $C_q \subset q$  of type  $(m + 1, m + 1)$  on  $q$ , intersecting at  $2w - \epsilon$  points on the line  $r_1 = \pi_1 \cap q$  and at  $w$  points on the line  $r_2 = \pi_2 \cap q$ . Moreover  $\deg(R \cap C'') = m + 1$ .

- Let  $k = 0$  and  $v = 2$ . In this case we have that  $0 \leq \epsilon < w$ . Let  $X$  be a smooth rational normal 3-fold in  $\mathbb{P}^5$  and let  $\pi$  be a plane of  $X$ . Let  $D \subset \pi$  be a plane curve of degree  $0 \leq \deg D = \epsilon + 1 < w + 1$  in  $\pi$ . With the same argument used in case  $k = v = 2$  one can prove that the general divisor in the linear system  $|\mathcal{I}_{D|X}(w + 1)|$  is a smooth surface  $S$  of degree  $s = 3w + 3$ . Let  $L$  be a general hyperplane section of  $X$  containing  $\pi$ , i.e. the union of a smooth quadric surface  $Q$  and  $\pi$  meeting along a line  $r$ . Let  $C'$  be the linked curve to  $D$  by the intersection  $S \cap L$ , i.e.  $C'$  is the union of a curve  $C'_Q \subset Q$  of type  $(w + 1, w + 1)$  and of a plane curve  $C'_\pi \subset \pi$  of degree  $w - \epsilon$ , meeting at  $w - \epsilon$  points. Let us consider the linear system  $|\mathcal{I}_{C'|S}(m + 1)|$ . By Bertini's Theorem the generic curve  $C$  in the movable part of this linear system is smooth of degree  $d$ . Moreover  $C$  lies on a smooth surface  $S$  of degree  $s = 3w + 3$  and it has the required numerical characters, as one may easily verify using  $p_a(C') = w^2 + \frac{1}{2}(w - \epsilon - 1)(w - \epsilon - 2) + w - \epsilon - 1$  (computed by Noether's formula) and  $\deg(R \cap C') = w + 1$  in the genus formula (3.13).

- Let  $k = 0$  and  $v = 1$ . Again we have  $0 \leq \epsilon < w$ . Let  $X$ ,  $\pi \subset X$  and  $p_1$  as in the case  $k = 2$  and  $v = 1$ . Let  $D \subset \pi$  be a smooth plane curve of degree  $\epsilon + 1$ . With the same argument used in the case  $k = 2$  and  $v = 1$  we find an irreducible smooth surface  $S$  in the movable part of a general divisor in the linear system  $|\mathcal{I}_{D \cup p_1|X}(w + 1)|$ . Let  $L = \pi \cup Q$  be a general hyperplane section of  $X$  containing the plane  $\pi$ , which intersects the plane  $p_1$  in a line  $r_1$  of type  $(1, 0)$ .  $C'$  is the linked curve to  $D$  by the intersection  $S \cap L$ ; therefore  $C'$  is the union of a curve  $C'_Q \subset Q$  of type  $(w, w + 1)$  and of a plane curve  $C'_\pi \subset \pi$  of degree  $w - \epsilon$ , meeting at  $w - \epsilon$  points. The generic curve  $C$  in the movable part of this linear system  $|\mathcal{I}_{C'|S}(m + 1)|$  is smooth of degree  $d$ . Moreover  $C$  lies on a smooth surface  $S$  of degree  $s = 3w + 2$  and it has the required numerical characters as one may easily verify using formula (3.13). In this case  $C''$  is the union of  $C'$  with a plane curve of degree  $m + 1$  on  $p_1$  meeting at  $w + 1$  points, and  $\deg(R \cap C'') = w + 1$ .

- Let  $k = 0$  and  $v = 0$ . We have  $0 \leq \epsilon < w$ . Let  $X$ ,  $\pi \subset X$  and  $L = \pi \cup Q$  as in the previous case. Let  $q \sim H - R$  be an other smooth quadric surface contained in  $X$  which intersects  $Q$  along a line  $r$  of type  $(0, 1)$  and  $\pi$  along a line  $s$  of type  $(1, 0)$ . Let  $D \subset \pi$  be a smooth plane curve of degree  $\epsilon$  (when  $\deg D = 1$  we suppose that  $D$  does not coincide with the line  $s$ ). With the same argument used in case  $k = 2$  and  $v = 0$  we find an irreducible smooth surface  $S$  is the movable part of a general divisor in the linear system  $|\mathcal{I}_{D \cup q|X}(w + 1)|$ .  $C'$  is the linked curve to  $D$  by the intersection  $S \cap L$ ; therefore  $C'$  is the union of a curve

$C'_Q \subset Q$  of type  $(w+1, w)$  and of a plane curve  $C'_\pi \subset \pi$  of degree  $w - \epsilon - 1$ , meeting at  $w - \epsilon - 1$  points. The generic curve  $C$  in the movable part of this linear system  $|\mathcal{I}_{C'|S}(m+1)|$  is smooth of degree  $d$ . Moreover  $C$  lies on a smooth surface  $S$  of degree  $s = 3w + 1$  and it has the required numerical characters as one may easily verify by using formula (3.13). In this case  $C''$  is the union of  $C'$  with a curve of type  $(m+1, m+1)$  on  $q$  meeting at  $w+1$  points along the line  $r$  and at  $w - \epsilon - 1$  points along the line  $s$ , and  $\deg(R \cap C'') = w + m + 1$ .

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